COVARIANT GEOMETRIC PREQUANTIZATION OF FIELDS

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A geometric prequantization formula for the Poisson-Gerstenhaber bracket of forms found within the De Donder-Weyl Hamiltonian formalism earlier is presented. The related aspects of covariant geometric quantization of field theories are sketched.

The use of the methods of geometric quantization[1] in field theory is severely limited by the infinite dimensionality of the standard field theoretic canonical Hamiltonian formalism because of the difficulties of substantiation of formal generalizations of geometric constructions to infinite dimensions.

We suggest an approach based on the *De Donder-Weyl* (DW) Hamiltonian form of field equations

$$\partial_\mu \phi^a(x) = \partial H/\partial p^\mu_a, \quad \partial_\mu p^\mu_a(x) = \partial H/\partial \phi^a, \tag{1}$$

where $p^\mu_a := \partial L/\partial \phi^a_{,\mu}$, $H(\phi^a, p^\mu_a, x^\nu) := \phi^a_{,\mu} p^\mu_a - L$, $L(\phi^a, \phi^a_{,\mu}, x^\nu)$ is a Lagrangian density, which incorporates the field dynamics into a finite dimensional *polymomentum phase space*: (p^μ_a, ϕ^a, x^μ) and requires no space+time splitting. The Poisson bracket for the above DW formulation is defined on differential forms which represent dynamical variables[2]. The basic structure is the *polysymplectic form* $\Omega := d\phi^a \wedge dp^\mu_a \wedge \omega_\mu$ ($\omega_\mu := \partial_\mu \rfloor (dx^1 \wedge ... \wedge dx^n)$) which maps ("horizontal") f-forms F to ("vertical") multivectors X_F of degree $(n-f)$: $X_F \rfloor \Omega = dF$. The map exists for the so-called *Hamiltonian forms* the space of which is closed with respect to the *co-exterior product*: $F \bullet G := *^{-1}(*F \wedge *G)$. The bracket

$$\{\![F, G]\!\} := (-)^{n-f} \pounds_{X_F}(G), \tag{2}$$

where $\pounds_{X_F} := [X_F, d] = X_F \circ d - (-)^{n-f} d \circ X_F$, equips the space of Hamiltonian forms with a *Gerstenhaber algebra* structure[2,3]:

$$\begin{gathered}\{\![F, G]\!\} = -(-1)^{g_1 g_2} \{\![G, F]\!\}, \\ (-1)^{g_1 g_3} \{\![F, \{\![G, K]\!\}]\!\} + (-1)^{g_1 g_2} \{\![G, \{\![K, F]\!\}]\!\} + (-1)^{g_2 g_3} \{\![K, \{\![F, G]\!\}]\!\} = 0, \\ \{\![F, G \bullet K]\!\} = \{\![F, G]\!\} \bullet K + (-1)^{g_1 (g_2+1)} G \bullet \{\![F, K]\!\},\end{gathered} \tag{3}$$

where $g_1 = n - f - 1$, $g_2 = n - g - 1$, $g_3 = n - k - 1$.

How to quantize fields using the above generalization of Poisson brackets? Our recent work on "precanonical quantization"[4,5] based on heuristic quantization of a small subalgebra of (3) leads to a generalization of quantum theoretic formalism in which the polymomenta operators are $\hat{p}^\mu_a = -i\hbar\kappa\gamma^\mu \frac{\partial}{\partial \phi^a}$ and the covariant "multi-temporal" analogue of the Schrödinger equation for $\Psi = \Psi(\phi^a, x^\mu)$ reads

$$i\hbar\kappa\gamma^\mu \partial_\mu \Psi = \widehat{H}\Psi, \tag{4}$$

where the constant $\kappa \sim \frac{1}{[\mathtt{length}]^{(n-1)}}$ is of the ultra-violet cutoff scale and, for the scalar field, $\widehat{H} = -\frac{1}{2}\hbar^2\kappa^2\partial^2_{\phi\phi} + V(\phi)$. It is notable that Eq. (4) enables us to derive the standard functional differential Schrödinger equation in quantum field theory[8].

The main result of the present contribution is a generalization of a cornerstone of geometric quantization, the prequantization formula, to the present framework[6]: the operator

$$\hat{F} = i\hbar \pounds_{X_F} + (X_F \rfloor \Theta) \bullet + F\bullet, \tag{5}$$

with $d\Theta := \Omega$, is a prequantization of a Hamiltonian form F in the sense that

$$[\hat{F_1}, \hat{F_2}] = -i\hbar \widehat{\{\![F_1, F_2]\!\}}, \tag{6}$$

where $[A, B] = A \circ B - (-1)^{\deg A \deg B} B \circ A$ is a graded commutator. The operator (5) is nonhomogeneous: the degree of the first term is $(n - f - 1)$ while the degree of the other two is $(n - f)$. Therefore, the first two terms can be viewed as a covariant derivative ∇_{X_F} corresponding to a superconnection[7]; then the polysymplectic form can be viewed as the curvature of the superconnection. Prequantum operators (5) can act on prequantum Hilbert space of nonhomogeneous differential forms $\Psi(\phi^a, p^\nu_a, x^\nu) = \sum_{p=0}^n \psi_{\mu_1 \ldots \mu_p} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p}$ with the scalar product $\langle \Psi, \Psi \rangle (x) = \sum_{p=0}^n \int (\overline{\psi}_{\mu_1 \ldots \mu_p} \psi^{\mu_1 \cdots \mu_p})$Vol, where Vol:$= \prod_{a=1}^m d\phi^a \wedge dp^1_a \wedge dp^2_a \wedge \ldots \wedge dp^n_a$ (no summation over index a here!). However, it is more suitable to Cliffordize the above expressions according to the rule $\omega_\mu \bullet = \kappa^{-1} \gamma_\mu$. Then the Cliffordized prequantum operators (5) can act on spinor wave functions $\Psi(\phi^a, p^\nu_a, x^\nu)$ with the positive definite scalar product $\langle \Psi, \Psi \rangle = \int \int_\sigma \overline{\Psi} \gamma^\mu \Psi$ Vol$\wedge \omega_\mu$, where σ is a space-like hypersurface. Using the "vertical polarization" with $\Psi = \Psi(\phi^a, x^\nu)$ one can derive from (5) the operators already known in precanonical quantization[4,5]. Thus, the structures of DW theory[2] naturally lead to the notions of Clifford/spinor bundles and superconnections[7] as a framework of generalizing the techniques of geometric quantization[1] to field theory, requiring neither a space+time splitting nor an infinite dimensional geometry.

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